1 1. Introduction

Let $p(x)$ be a polynomial of degree $n \geq 2$ with n distinct real roots $r_1 < r_2 < \cdots < r_n$. Such a polynomial is called hyperbolic. Let $x_1 < x_2 < \cdots < x_{n-1}$ be the critical points of p, and define the ratios $\sigma_k = \dfrac{x_k - r_k}{r_{k+1} - r_k}, k = 1, 2, ..., n-1$. $(\sigma_1, ..., \sigma_{n-1})$ is called the *ratio vector* of p, and σ_k is called the kth ratio. Ratio vectors were first discussed in [5] and in [1], where the inequality $\dfrac{1}{n-k+1} < \sigma_k < \dfrac{k}{k+1}, k = 1, 2, ..., n-1$ was derived. In a similar fashion, one can define ratios for polynomial like functions of the form $p(x) = (x-r_1)^{m_1} \cdots (x-r_N)^{m_N}$, where $m_1, ..., m_N$ are given positive real numbers and $r_1 < r_2 < \cdots < r_N$(see [4]).

In this paper we want to discuss the extension of the notion of ratios to polynomials with *complex* roots. Thus we let $p(z)$ be a polynomial of degree $n \geq 2$ with n distinct complex roots $w_1, ..., w_n$ and critical points $z_1, ..., z_{n-1}$. Numerous papers have investigated the relation between the roots and critical points of a polynomial. The focus of this paper is to investigate that relation in the form of the complex ratios $\sigma_k = \dfrac{z_k - w_k}{w_{k+1} - w_k}, k = 1, 2, ..., n-1$. The main problem is in defining the ratios when there is no natural ordering of roots and critical points as with all real roots. We have to order the $\{w_k\}$ somehow and then determine which $\{z_k\}$ are associated with w_k and w_{k+1}. We use the real parts of the $\{w_k\}$ and the $\{z_k\}$ to do this. For the rest of the paper we concentrate solely on the case $n = 3$, which is already fairly nontrivial. We do not define the ratios in the case when two roots or critical points have equal real parts(unless the critical points are identical). One could certainly extend the definition to those cases, but the ratios will not be continuous function of the roots. Our definition does extend the definition of the ratios when p is hyperbolic and the ratios are continuous functions of the roots when the roots are all real. For cubic hyperbolic polynomials, the inequality $\dfrac{1}{n-k+1} < \sigma_k < \dfrac{k}{k+1}$ implies that $\dfrac{1}{3} < \sigma_1 < \dfrac{1}{2}$ and $\dfrac{1}{2} < \sigma_2 < \dfrac{2}{3}$. For complex ratios, we derive separate and sharp upper and lower bounds on the real and imaginary parts, and modulus, of each ratio(see Theorems 1 and 2). For cubic hyperbolic polynomials, it is immediate that $\sigma_1 < \sigma_2$. In the complex case we prove that $\operatorname{Re}\sigma_1 \leq \operatorname{Re}\sigma_2$(Theorem 3). Indeed, one can have $\sigma_1 = \sigma_2$(see Theorem 4). Finally, we show that the ratios are real if and only if the roots of p are collinear(Theorem 5).

2 2. Main Results

Let

$$p(w) = (w - w_1)(w - w_2)(w - w_3),$$

where we assume that $\operatorname{Re} w_1 < \operatorname{Re} w_2 < \operatorname{Re} w_3$. Now the critical points of p are $\frac{1}{3}\left(w_1 + w_2 + w_3 \pm \sqrt{w_1^2 + w_2^2 + w_3^2 - w_1 w_3 - w_1 w_2 - w_2 w_3}\right)$, where $\sqrt{z}$ is the principal branch of the square root function, analytic everywhere except on the *nonpositive* real axis, which we denote by Γ. Note that $\operatorname{Re}\sqrt{z} \geq 0$ and $\sqrt{z^2} = z$ if $\operatorname{Re} z \geq 0$. We also assume that if the critical points are not identical, then they cannot have equal real parts. In other words, we assume that

$\operatorname{Re}\sqrt{w_1^2 + w_2^2 + w_3^2 - w_1 w_3 - w_1 w_2 - w_2 w_3} \neq 0$ unless $w_1^2 + w_2^2 + w_3^2 - w_1 w_3 - w_1 w_2 - w_2 w_3 = 0$. Denote the critical points by z_1 and z_2, where $z_1 = z_2$, or $\operatorname{Re} z_1 < \operatorname{Re} z_2$ if $z_1 \neq z_2$. We define the ratios

$$\sigma_1 = \frac{z_1 - w_1}{w_2 - w_1}, \sigma_2 = \frac{z_2 - w_2}{w_3 - w_2} \tag{(1)}$$

(σ_1, σ_2) is called the *ratio vector* of p. One can give a geometric interpretation for the ratios as follows. First, if $p \in \pi_3$ has *noncollinear* zeros, let T be the triangle whose vertices are w_1, w_2, w_3. Let E be the midpoint ellipse, that is, the ellipse tangent to T at the midpoints of its sides. Then it is well known that the zeros of p' are the foci, z_1 and z_2, of E. Let θ_1 denote the angle between $\overrightarrow{w_1 z_1}$ and $\overrightarrow{w_1 w_2}$, and let θ_2 denote the angle between $\overrightarrow{w_2 z_2}$ and $\overrightarrow{w_2 w_3}$. Then $\theta_1 = \arg \sigma_1$ and $\theta_2 = \arg \sigma_2$, and thus each ratio represents an angle between a line segment of the circumscribed triangle of E and a line segment connecting a vertex to one of the foci of E.

Using our definition, neither σ_1 nor σ_2 will be a continuous function of w_1, w_2, and w_3, though they are continuous on an open subset of $C^3 - \{w_1, w_2, w_3 : w_i = w_j \text{ for some } i \neq j\}$ and in particular at any point (w_1, w_2, w_3) where all of the $\{w_k\}$ are real. Clearly, if we translate the roots of p, the ratios σ_1 and σ_2 do not change. Thus we may assume that

$$w_1 + w_3 = 0 \tag{(2)}$$

which implies that $\operatorname{Re} w_1 < 0 < \operatorname{Re} w_3$. Note that $\operatorname{Re}\sqrt{w_3} > 0$. The critical points of p are then $\frac{1}{3}\left(w_2 \pm \sqrt{3w_3^2 + w_2^2}\right)$. The assumption that if the critical points are not identical, then they cannot have equal real parts now takes the form

$$3w_3^2 + w_2^2 \neq 0 \Rightarrow \operatorname{Re}\sqrt{3w_3^2 + w_2^2} \neq 0$$

If $3w_3^2 + w_2^2 \neq 0$, then by our choice of the branch of $\sqrt{z}$, $\operatorname{Re}\sqrt{3w_3^2 + w_2^2} > 0$, which implies that $\operatorname{Re}\left(w_2 - \sqrt{3w_3^2 + w_2^2}\right) < \operatorname{Re}\left(w_2 + \sqrt{3w_3^2 + w_2^2}\right)$. Thus we have

$$z_1 = \frac{1}{3}\left(w_2 - \sqrt{3w_3^2 + w_2^2}\right), \; z_2 = \frac{1}{3}\left(w_2 + \sqrt{3w_3^2 + w_2^2}\right)$$

Also, $\operatorname{Re}\sqrt{3w_3^2 + w_2^2} > 0 \iff 3w_3^2 + w_2^2 \notin \Gamma$. That leads to the following.

Definition: We say that (w_2, w_3) is an admissible pair if w_2 and w_3 satisfy $3w_3^2 + w_2^2 \notin \Gamma$, $w_2 + w_3 \neq 0$, $\operatorname{Re} w_2 < \operatorname{Re} w_3$, and $0 < \operatorname{Re} w_3$. A region in C^2 consisting of only admissible pairs is also called admissible.

Note that the ratios are not defined, say, when $w_1 = -1, w_2 = ti, w_3 = 1$, $|t| > \sqrt{3}$, since in that case $\operatorname{Re}\sqrt{3w_3^2 + w_2^2} = 0$, which implies that $\operatorname{Re} z_1 = \operatorname{Re} z_2$, but $z_1 \neq z_2$. Let

$$w = \frac{w_2}{w_3}. \tag{(3)}$$

We shall express σ_1 and σ_2 as analytic functions of w. We then derive bounds on the real part, imaginary part, and modulus of the ratios and also some relations between the ratios. By (1) and (2), $\sigma_1 = \dfrac{z_1 + w_3}{w_2 + w_3} = \dfrac{1}{3}\dfrac{3z_1 + 3w_3}{w_2 + w_3} = \dfrac{1}{3}\dfrac{w_2 + 3w_3 - \sqrt{3w_3^2 + w_2^2}}{w_2 + w_3} = \dfrac{1}{3}\dfrac{w_2 + 3w_3 - \sqrt{w_3^2\left(3 + \frac{w_2^2}{w_3^2}\right)}}{w_2 + w_3}$. In general, $\sqrt{w_3^2\left(3 + \frac{w_2^2}{w_3^2}\right)} = \pm\sqrt{w_3^2}\sqrt{3 + \frac{w_2^2}{w_3^2}} = w_3\sqrt{3 + \frac{w_2^2}{w_3^2}}$ or $-w_3\sqrt{3 + \frac{w_2^2}{w_3^2}}$ and thus $\sigma_1 = f_1(w_2, w_3)$ or $\sigma_1 = f_2(w_2, w_3)$, where $f_1(w_2, w_3) = \dfrac{1}{3}\dfrac{\frac{w_2}{w_3} + 3 - \sqrt{3 + \frac{w_2^2}{w_3^2}}}{\frac{w_2}{w_3} + 1}$ and $f_2(w_2, w_3) = \dfrac{1}{3}\dfrac{\frac{w_2}{w_3} + 3 + \sqrt{3 + \frac{w_2^2}{w_3^2}}}{\frac{w_2}{w_3} + 1}$. Now $\dfrac{1}{3}\dfrac{w_2 + 3w_3 - \sqrt{3w_3^2 + w_2^2}}{w_2 + w_3}$ must be an analytic function of w_2 and w_3 in any admissible region. f_1 and f_2 are also analytic functions of w_2 and w_3 in any admissible region with the additional assumption that

$$3 + \frac{w_2^2}{w_3^2} \notin \Gamma. \tag{(4)}$$

Since $3 + \frac{w_2^2}{w_3^2} \neq 0$, it follows that $\dfrac{1}{3}\dfrac{w_2 + 3w_3 - \sqrt{3w_3^2 + w_2^2}}{w_2 + w_3}$ must equal $f_1(w_2, w_3)$ or $f_2(w_2, w_3)$. Now $f_1(0, 1) = 1 - \frac{1}{3}\sqrt{3}$, while $f_2(0, 1) = 1 + \frac{1}{3}\sqrt{3}$. But $w_2 = 0$ and $w_3 = 1$ yields the polynomial $p(z) = z(z^2 - 1)$, and it is easy to check that $\sigma_1 = 1 - \frac{1}{3}\sqrt{3}$. It then follows that $\dfrac{1}{3}\dfrac{w_2 + 3w_3 - \sqrt{3w_3^2 + w_2^2}}{w_2 + w_3} = \dfrac{1}{3}\dfrac{\frac{w_2}{w_3} + 3 + \sqrt{3 + \frac{w_2^2}{w_3^2}}}{\frac{w_2}{w_3} + 1}$. Using (3) we have $\sigma_1 = \dfrac{1}{3}\dfrac{w + 3 - \sqrt{3 + w^2}}{w + 1}$. Now let

$$E = \left\{w : \operatorname{Re} w = 0, |\operatorname{Im} w| \geq \sqrt{3}\right\}$$

and let

$$D_1 = C^2 - E - \{w : w = -1\}, D_2 = C^2 - E - \{w : w = 1\}.$$

Note that $w \in C^2 - E \iff (w_2, w_3)$ satisfies (4). Then

$$\sigma_1 = \frac{1}{3}\frac{w + 3 - \sqrt{3 + w^2}}{w + 1}, w \in D_1.$$

In a similar fashion one can show that

$$\sigma_2 = \frac{1}{3}\frac{-2w + \sqrt{3 + w^2}}{1 - w}, w \in D_2.$$

This expression for σ_2 also follows from the equation

$$(1 - \sigma_1)\,\sigma_2 = \frac{1}{3} \tag{(5)}$$

(5) is easy to prove and the proof is exactly the same as for the case when p has three distinct real roots (see [1] or [3]). It is now convenient to define the following analytic extensions of σ_1 to $w = -1$ and of σ_2 to $w = 1$, respectively.

$$f(w) = \begin{cases} \frac{1}{3}\frac{w + 3 - \sqrt{3 + w^2}}{w + 1} = \sigma_1 & \text{if } w \in D_1 \\ \frac{1}{2} & \text{if } w = -1 \end{cases}$$

and

$$g(w) = \begin{cases} \frac{1}{3}\frac{-2w + \sqrt{3 + w^2}}{1 - w} = \sigma_2 & \text{if } w \in D_2 \\ \frac{1}{2} & \text{if } w = 1 \end{cases}$$

Since $\lim_{w \to -1} \frac{1}{3}\frac{w + 3 - \sqrt{3 + w^2}}{w + 1} = \frac{1}{2}$ and $\lim_{w \to 1} \frac{1}{3}\frac{-2w + \sqrt{3 + w^2}}{1 - w} = \frac{1}{2}$, f and g are each analytic in the region

$$D = C^2 - E.$$

We can now replace (5) by

$$(1 - f(w))g(w) = \frac{1}{3}, w \in D \tag{(6)}$$

Note that f does *not* extend to be continuous on $\partial\,(D)$ because of the discontinuity of $\sqrt{3 + w^2}$ when $3 + w^2 \in \Gamma$. Also, for $w \in \partial\,(D)\,, f(w)$ does not yield σ_1 and $g(w)$ does not yield σ_2. Now

$$w \in \partial\,(D) \iff w = ti, |t| \geq \sqrt{3}.$$

Then $w_1 = -w_3, w_2 = tiw_3$, and $p(z) = (z^2 - w_3^2)(z - itw_3)$. If $\operatorname{Im} w_3 \neq 0$, then the ratios are defined, and a simple computation shows that

$$\sigma_1 = \frac{1}{3}\frac{it + i\sqrt{t^2 - 3} + 3}{it + 1}, w \in \partial\,(D) \tag{(7)}$$

One can also compute σ_2 using (5), but we shall not require that here.

Notation: We write $\sigma_1 = \sigma_1(w)$ or $\sigma_2 = \sigma_2(w)$ if (σ_1, σ_2) is the ratio vector of $p(w) = (w - w_1)(w - w_2)(w - w_3)$ with $w_1 + w_3 = 0, \operatorname{Re} w_1 < 0 < \operatorname{Re} w_3$, and $w = \dfrac{w_2}{w_3}$

We should note here that not every $w \in D$ satisfies $w = \dfrac{w_2}{w_3}$ for some admissible pair (w_2, w_3). For example, $w = 2$ cannot occur since $w_2 = 2w_3 \Rightarrow \operatorname{Re} w_2 > \operatorname{Re} w_3$. Of course the bounds we derive for $w \in D \cup \partial(D)$ then apply to the subset of values of w which can arise from admissible pairs. In addition, there are admissible pairs (w_2, w_3) such that $w\partial(D)$, such as $w_2 = 2i, w_3 = 1$. This is not a problem since the bounds we derive below are for $w \in D \cup \partial(D)$. Finally, the ratios themselves are not defined when $w = 1$ or $w = -1$(else the w_k are not distinct). The real and imaginary parts of f and of g are each harmonic functions, and we want to apply the Maximum–Minimum Principle for harmonic functions to find bounds on the real and imaginary parts of σ_1 and σ_2. Since D is unbounded, we shall require the following special case of the Maximum–Minimum Principle for possibly unbounded domains(see [2], page 8, Corollary 1.10]).

Proposition 1: Let u be a real–valued harmonic function in a domain D in R^2 and suppose that

$$\limsup_{k \to \infty} u(a_k) \leq M$$

for every sequence $\{a_k\}$ in D converging to a point in $\partial(D)$ or to ∞. Then $u \leq M$ on D.

Remark: As noted in [2], Proposition 1 remains valid if "lim sup" is replaced by "lim inf" and the inequalities are reversed.

We also need the following Local Maximum–Minimum Principle for harmonic functions for possibly unbounded domains(see [2], page 23) to prove the sharpness of our bounds on the real and imaginary parts of σ_1 and σ_2. One can prove these bounds directly, but that involves a two variable optimization problem. Using the Maximum–Minimum Principle reduces it to a one variable optimization problem.

Proposition 2: Let u be a real–valued harmonic function in a domain D in R^2 and suppose that u has a local maximum(or minimum) in D. Then u is constant.

First we require the following lemmas.

Lemma 1: (A) The equation $4t\sqrt{t^2 - 3} - 5t^2 + 3 = 0$ has no real solutions.

(B) The equation $4t\sqrt{t^2 - 3} + 5t^2 - 3 = 0$ has no real solutions.

Proof: $4t\sqrt{t^2 - 3} = 5t^2 - 3 \Rightarrow 16t^2\left(t^2 - 3\right) - (5t^2 - 3)^2 = 0 \Rightarrow -9\left(t^2 + 1\right)^2 = 0$, which has no real solutions. That proves (A), and (B) follows in a similar fashion.

Lemma 2: (A)The only real solution of the equation $t^3 - 7t - 2\left(t^2 - 1\right)\sqrt{t^2 - 3} = 0$ is $t = -2$.

(B) The only real solution of the equation $t^3 - 7t + 2\left(t^2 - 1\right)\sqrt{t^2 - 3} = 0$ is $t = 2$.

Proof: $t^3-7t=2\left(t^2-1\right)\sqrt{t^2-3}\Rightarrow(t^3-7t)^2-4\left(t^2-1\right)^2(t^2-3)=0\Rightarrow$ $-3\left(t-2\right)\left(t+2\right)\left(t^2+1\right)^2=0$. $t=-2$ is a solution of the given equation, but not $t=2$. That proves (A), and (B) follows in a similar fashion.

Theorem 1: Let $p(w)=(w-w_1)(w-w_2)(w-w_3)$,with $\operatorname{Re}w_1<\operatorname{Re}w_2<\operatorname{Re}w_3$. Let z_1 and z_2 be the critical points of p, where $z_1=z_2$ or $\operatorname{Re}z_1<\operatorname{Re}z_2$ if $z_1\neq z_2$. Let $\sigma_1=\dfrac{z_1-w_1}{w_2-w_1}$. Then

(A) $0<\operatorname{Re}\sigma_1<\dfrac{2}{3}$ and the inequality is sharp in that there are w_1,w_2, and w_3 satisfying the hypotheses above and such that $\operatorname{Re}\sigma_1$ can be made arbitrarily close to 0 or arbitrarily close to $\dfrac{2}{3}$.

(B) $-\dfrac{1}{3}\leq\operatorname{Im}\sigma_1\leq\dfrac{1}{3}$.

(C) $\operatorname{Im}\sigma_1=\dfrac{1}{3}\iff$ the roots of p have the form $\pm i\left(z_0+C\right)$ and $2\left(z_0+C\right)$, where $\operatorname{Im}z_0<0$, $0<\operatorname{Re}z_0<-\dfrac{1}{2}\operatorname{Im}z_0$, and C is an arbitrary constant.

(D) $\operatorname{Im}\sigma_1=-\dfrac{1}{3}\iff$ the roots of p have the form $\pm i\left(z_0+C\right)$ and $2\left(z_0+C\right)$, where $\operatorname{Im}z_0>0$, $0<\operatorname{Re}z_0<\dfrac{1}{2}\operatorname{Im}z_0$, and C is an arbitrary constant.

(E) $|\sigma_1|\leq\dfrac{2}{3}$

Proof: While it is not necessary for f to extend to be continuous on $\partial\left(D\right)$ to apply Proposition 1, we must show that $0<f(w)<\dfrac{2}{3}$ for $w\in D\cup\partial\left(D\right)$ since σ_1 can arise for $w\in\partial\left(D\right)$. First we consider the behavior of f at ∞. $\lim\limits_{w\to\infty}f(w)=\lim\limits_{w\to 0}f(1/w)=\dfrac{1}{3}\lim\limits_{w\to 0}\dfrac{1+3w\pm\sqrt{3w^2+1}}{w+1}=0$ or $\dfrac{2}{3}$ depending upon whether $w\to 0$ through $\operatorname{Re}w>0$ or $\operatorname{Re}w<0$. Thus by Proposition 1, $\limsup\limits_{k\to\infty}\operatorname{Re}f(a_k)\leq\dfrac{2}{3}$, $\liminf\limits_{k\to\infty}\operatorname{Re}f(a_k)\geq 0$, $\limsup\limits_{k\to\infty}\operatorname{Im}f(a_k)\leq\dfrac{1}{3}$, and $\liminf\limits_{k\to\infty}\operatorname{Im}f(a_k)\geq-\dfrac{1}{3}$ for any sequence $\{a_k\}$ in D converging to ∞. We now show that $0\leq\operatorname{Re}f\leq\dfrac{2}{3}$ and $-\dfrac{1}{3}\leq\operatorname{Im}f\leq\dfrac{1}{3}$ as w *approaches* any point $z\in\partial\left(D\right)$. As w approaches $z\in\partial\left(D\right)$, $\sqrt{3+w^2}$ approaches $\pm\sqrt{3-t^2}=\pm i\sqrt{t^2-3}$. Thus $\dfrac{1}{3}\dfrac{w+3-\sqrt{3+w^2}}{w+1}$ approaches $\dfrac{1}{3}\dfrac{ti+3\pm i\sqrt{t^2-3}}{ti+1}$. Note that $\dfrac{1}{3}\dfrac{ti+3+i\sqrt{t^2-3}}{ti+1}=\sigma_1(w), w\in\partial\left(D\right)$ by (7). Thus by finding the maximum and minimum of $\operatorname{Re}\dfrac{1}{3}\dfrac{ti+3\pm i\sqrt{t^2-3}}{ti+1}$ and $\operatorname{Im}\dfrac{1}{3}\dfrac{ti+3\pm i\sqrt{t^2-3}}{ti+1}, |t|\geq\sqrt{3}$, we are finding the maximum and minimum of $\operatorname{Re}f(w)$ and of $\operatorname{Im}f(w)$ as w approaches $\partial\left(D\right)$, and the maximum and minimum of $\operatorname{Re}\sigma_1$ and of $\operatorname{Im}\sigma_1$ for

$w \in \partial(D)$. Now

$$\frac{1}{3}\frac{ti+3+i\sqrt{t^2-3}}{ti+1} = u_1(t)+iv_1(t), \frac{1}{3}\frac{ti+3-i\sqrt{t^2-3}}{ti+1} = u_2(t)+iv_2(t)$$

where

$$u_1(t) = \frac{1}{3}\frac{t^2+3+t\sqrt{t^2-3}}{t^2+1}, u_2(t) = \frac{1}{3}\frac{t^2+3-t\sqrt{t^2-3}}{t^2+1} \tag{(8)}$$

and

$$v_1(t) = \frac{1}{3}\frac{-2t+\sqrt{t^2-3}}{t^2+1}, v_2(t) = \frac{1}{3}\frac{-2t-\sqrt{t^2-3}}{t^2+1} \tag{(9)}$$

$u_1'(t) = -\frac{1}{3}\frac{4t\sqrt{t^2-3}-5t^2+3}{\sqrt{t^2-3}\,(t^2+1)^2}$ and $u_2'(t) = -\frac{1}{3}\frac{4t\sqrt{t^2-3}+5t^2-3}{\sqrt{t^2-3}\,(t^2+1)^2}$. By Lemma 1, u_1' and u_2' have no real roots, and hence u_1 and u_2 have no real critical points. Now $u_1(\sqrt{3}) = u_1(-\sqrt{3}) = u_2(\sqrt{3}) = u_2(-\sqrt{3}) = \frac{1}{2}$, $\lim_{t\to\infty} u_1(t) = \lim_{t\to-\infty} u_2(t) = \frac{2}{3}$, and $\lim_{t\to-\infty} u_1(t) = \lim_{t\to\infty} u_2(t) = 0$. Thus $0 \le u_1(t), u_2(t) \le \frac{2}{3}$ for $|t| \ge \sqrt{3}$, which implies that $0 \le \operatorname{Re}\frac{1}{3}\frac{ti+3\pm i\sqrt{t^2-3}}{ti+1} \le \frac{2}{3}$for $|t| \ge \sqrt{3}$. It follows that $\limsup_{k\to\infty}\operatorname{Re} f(a_k) \le \frac{2}{3}$ and $\liminf_{k\to\infty}\operatorname{Re} f(a_k) \ge 0$ for any sequence $\{a_k\}$ in D converging to $\partial(D)$. By Proposition 1, $0 \le f(w) \le \frac{2}{3}$ for $w \in D$. As noted above, the same proof shows that $0 \le \operatorname{Re} f \le \frac{2}{3}$ for $w \in \partial(D)$. By Proposition 2, $0 < \operatorname{Re} f < \frac{2}{3}$ for $w \in D$. It also follows easily that u_2 is increasing for $t \le -\sqrt{3}$ and decreasing for $t \ge \sqrt{3}$, which implies that $u_2(t) \ne 0$ and $u_2(t) \ne \frac{2}{3}$ for $|t| \ge \sqrt{3}$. Since $u_2(t) = \operatorname{Re} f(w), w = ti, |t| \ge \sqrt{3}, 0 < \operatorname{Re} f < \frac{2}{3}$ for $w \in \partial(D)$. That shows that $0 < \operatorname{Re}\sigma_1 < \frac{2}{3}$. To finish the proof of part (A), if $t > \sqrt{3}$, let $w_1 = -2t-i$, $w_2 = -t+2t^2 i$, and $w_3 = 2t+i$, while if $t < -\sqrt{3}$, let $w_1 = 2t+i$, $w_2 = t-2t^2 i$, and $w_3 = -2t-i$. In either case, $w = ti$ and $\operatorname{Im}\left(3w_3^2+w_2^2\right) = 12t-4t^3 \ne 0 \Rightarrow \operatorname{Re}\sqrt{3w_3^2+w_2^2} \ne 0$. Thus z_1 and z_2 have unequal real parts. Since $\operatorname{Re} w_1 < \operatorname{Re} w_2 < \operatorname{Re} w_3$ as well, the ratios are defined. Above we showed that

$$\sigma_1(w) = u_1(t)+iv_1(t), w = it, |t| \ge \sqrt{3} \tag{(10)}$$

Thus $\operatorname{Re}\sigma_1(w) = u_1(t)$. Since $\lim_{t\to\infty} u_1(t) = \frac{2}{3}$ and $\lim_{t\to-\infty} u_2(t) = 0$, we can make $\operatorname{Re}\sigma_1$ as close to 0 or $\frac{2}{3}$ by taking $|t|$ sufficiently large. That finishes the proof of part (A).

To prove part (B), $v_1'(t) = \dfrac{1}{3}\dfrac{2\sqrt{t^2-3}t^2 - 2\sqrt{t^2-3} - t^3 + 7t}{\sqrt{t^2-3}\,(t^2+1)^2}$ and $v_2'(t) = \dfrac{1}{3}\dfrac{2\sqrt{t^2-3}t^2 - 2\sqrt{t^2-3} + t^3 - 7t}{\sqrt{t^2-3}\,(t^2+1)^2}$. By Lemma 2, v_1 has one real critical point, $t = -2$ and v_2 has one real critical point, $t = 2$. Also, $v_1(\sqrt{3}) = -\frac{1}{6}\sqrt{3}$, $v_1(-\sqrt{3}) = \frac{1}{6}\sqrt{3}$, $v_1(-2) = \frac{1}{3}$, and $\lim_{t\to-\infty} v_1(t) = \lim_{t\to\infty} v_1(t) = 0$, while $v_2(\sqrt{3}) = -\frac{1}{6}\sqrt{3}$, $v_2(-\sqrt{3}) = \frac{1}{6}\sqrt{3}$, $v_2(2) = -\frac{1}{3}$, and $\lim_{t\to-\infty} v_2(t) = \lim_{t\to\infty} v_2(t) = 0$. Hence $-\frac{1}{3} \le v_1(t), v_2(t) \le \frac{1}{3}$ for $|t| \ge \sqrt{3}$. Arguing as earlier, by Proposition 1 that proves part (B).

To prove (C), suppose that $\operatorname{Im}\sigma_1 = \frac{1}{3}$. If $\sigma_1 = \sigma_1(w), w \in D$, then $\operatorname{Im} f(w) = \frac{1}{3}$, which cannot happen by Proposition 2. If $\sigma_1 = \sigma_1(w), w \in \partial(D)$, then $v_1(t) = \frac{1}{3}$ by (7). Now it follows easily that the only real solution of $v_1(t) = \frac{1}{3}$ is $t = -2$, and $t = -2 \Rightarrow w = -2i \Rightarrow w_3 = \frac{1}{2}iw_2, w_1 = -\frac{1}{2}iw_2$. The critical points of the coresponding p are $z = \frac{1}{2}w_2$ and $z = \frac{1}{6}w_2$, which have unequal real parts if $\operatorname{Re} w_2 \ne 0$. $\operatorname{Re} w_3 > 0 \Rightarrow -\frac{1}{2}\operatorname{Im} w_2 > 0 \Rightarrow \operatorname{Im} w_2 < 0$. Also, $\operatorname{Re} w_2 < \operatorname{Re} w_3 \Rightarrow \operatorname{Re} w_2 < -\frac{1}{2}\operatorname{Im} w_2$. If $\operatorname{Re} w_2 < 0$, then $z_1 = \frac{1}{2}w_2$ and $z_2 = \frac{1}{6}w_2 \Rightarrow \sigma_1 = \dfrac{\frac{1}{2}w_2 + \frac{1}{2}iw_2}{w_2 + \frac{1}{2}iw_2} = \frac{3}{5} + \frac{1}{5}i \Rightarrow \operatorname{Im}\sigma_1 \ne \frac{1}{3}$. Letting $z_0 = \frac{1}{2}w_2$, that yields roots of the form $\pm iz_0$ and $2z_0$, where $\operatorname{Re} z_0 > 0$ and $\operatorname{Re} z_0 < -\frac{1}{2}\operatorname{Im} z_0$. Since any translation of p yields the same ratios, the roots of p must have the form given in part (C). If the roots of p have the form given in part (C), then $z_1 = \frac{1}{6}w_2$ and $z_2 = \frac{1}{2}w_2$, which implies that $\sigma_1 = \dfrac{\frac{1}{6}w_2 + \frac{1}{2}iw_2}{w_2 + \frac{1}{2}iw_2} = \frac{1}{3} + \frac{1}{3}i \Rightarrow \operatorname{Im}\sigma_1 = \frac{1}{3}$. The proof of part (D) follows in a simialr fashion and we omit it.

Finally, to prove (E), note first that $f(w) = 0 \Rightarrow w + 3 - \sqrt{3+w^2} = 0 \Rightarrow (w+3)^2 - \left(3+w^2\right) = 6w + 6 = 0 \Rightarrow w = -1$, but $f(-1) = \frac{1}{2} \ne 0$. Thus f has no zero in D and by ([6], Theorem 13.12, page 294), $\log|f|$ is harmonic in D. We shall apply Proposition 1 to $\log|f|$. Since we showed earlier that $\lim_{w\to\infty} f(w) = 0$ or $\frac{2}{3}$, $\limsup_{k\to\infty} \log|f|(a_k) \le \log\frac{2}{3}$ for any sequence

$\{a_k\}$ in D converging to ∞. As w approaches $\partial(D)$, $9|f(w)|^2$ approaches $9\left[(u_1(t))^2+(v_1(t))^2\right]$ or $9\left[(u_2(t))^2+(v_2(t))^2\right]$, where $w=it, |t|\geq\sqrt{3}$. Now $9\left[(u_1(t))^2+(v_1(t))^2\right]=a(t)=2\dfrac{t^2+3+t\sqrt{t^2-3}}{t^2+1}$, and it follows easily that $a'(t)=2\dfrac{-4t\sqrt{t^2-3}+5t^2-3}{\sqrt{t^2-3}\,(t^2+1)^2}>0$ for all $t, |t|\geq\sqrt{3}$. Since $\lim\limits_{t\to\infty}a(t)=4$, $a(t)<4$ for all $t, |t|\geq\sqrt{3}$. $9\left[(u_2(t))^2+(v_2(t))^2\right]=b(t)=2\dfrac{t^2+3-t\sqrt{t^2-3}}{t^2+1}$. It also follows easily that $b'(t)=-2\dfrac{4t\sqrt{t^2-3}+5t^2-3}{\sqrt{t^2-3}\,(t^2+1)^2}<0$ for all $t, |t|\geq\sqrt{3}$. Since $\lim\limits_{t\to-\infty}b(t)=4$, $b(t)<4$ for all $t, |t|\geq\sqrt{3}$. Hence $|f(a_k)|^2\leq\dfrac{4}{9}$ for any sequence $\{a_k\}$ in D converging to $\partial(D)$, which implies that $\limsup\limits_{k\to\infty}\log|f(a_k)|\leq\dfrac{1}{2}\log\dfrac{4}{9}$. That proves that $|f(w)|\leq\dfrac{2}{3}, w\in D$, by Proposition 1. Note also that $9|\sigma_1|^2=9\left[(u_1(t))^2+(v_1(t))^2\right]$ for $w\in\partial(D)$. By what we just proved, $|\sigma_1(w)|\leq\dfrac{2}{3}, w\in\partial(D)$. That finishes the proof of part (E).

Theorem 2: Let $p(w)=(w-w_1)(w-w_2)(w-w_3)$,with $\operatorname{Re}w_1<\operatorname{Re}w_2<\operatorname{Re}w_3$. Let z_1 and z_2 be the critical points of p, where $z_1=z_2$ or $\operatorname{Re}z_1<\operatorname{Re}z_2$ if $z_1\neq z_2$. Let $\sigma_2=\dfrac{z_2-w_2}{w_3-w_2}$. Then

(A) $\dfrac{1}{3}<\operatorname{Re}\sigma_2<1$ and the inequality is sharp in that there are w_1, w_2, and w_3 satisfying the hypotheses above and such that $\operatorname{Re}\sigma_2$ can be made arbitrarily close to $\dfrac{1}{3}$ or arbitrarily close to 1.

(B) $-\dfrac{1}{3}\leq\operatorname{Im}\sigma_2\leq\dfrac{1}{3}$

(C) $\operatorname{Im}\sigma_2=\dfrac{1}{3}\iff$ the roots of p have the form $\pm iz$ and $2z$, where $\operatorname{Im}z<0$ and $0<\operatorname{Re}z<-\dfrac{1}{2}\operatorname{Im}z$

(D) $\operatorname{Im}\sigma_2=-\dfrac{1}{3}\iff$ the roots of p have the form $\pm iz$ and $2z$, where $\operatorname{Im}z>0$ and $\operatorname{Re}z>0$ and $0<\operatorname{Re}z<\dfrac{1}{2}\operatorname{Im}z$

(E) $|\sigma_2|\leq 1$

Proof: We proceed exactly as in the proof of Theorem1, working with $g(w)$ instead of with $f(w)$. Since $\lim\limits_{w\to\infty}f(w)=0$ or $\dfrac{2}{3}$, by (6), $\lim\limits_{w\to\infty}g(w)=\dfrac{1}{3}$ or 1. As w approaches $z\in\partial(D)$, $g(w)$ approaches $\dfrac{1}{3}\dfrac{-2ti\pm\sqrt{3-t^2}}{1-ti}=\dfrac{1}{3}\dfrac{-2ti\pm i\sqrt{t^2-3}}{1-ti}=1-u_1(t)+iv_1(t)$ or $1-u_2(t)+iv_2(t)$. Since we showed that $0<u_1(t)<\dfrac{2}{3}$, $0<u_2(t)<\dfrac{2}{3}, -\dfrac{1}{3}\leq v_1(t)\leq\dfrac{1}{3}$, and $-\dfrac{1}{3}\leq v_2(t)\leq\dfrac{1}{3}$ for

$|t| \geq \sqrt{3}$, it follows immediately that $\frac{1}{3} < \operatorname{Re}\sigma_2 < 1$ and $-\frac{1}{3} \leq \operatorname{Im}\sigma_2 \leq \frac{1}{3}$. The rest of parts (A) and (B) follow as in the proof of Theorem 1, parts (A) and (B). Parts (C) and (D) also follow as in the proof of Theorem 1 parts (C) and (D), and part (E) follows directly from Theorem 1, part (E) and (5).

Theorem 3: Let $p(w) = (w - w_1)(w - w_2)(w - w_3)$,with $\operatorname{Re} w_1 < \operatorname{Re} w_2 < \operatorname{Re} w_3$. Let z_1 and z_2 be the critical points of p, where $z_1 = z_2$ or $\operatorname{Re} z_1 < \operatorname{Re} z_2$ if $z_1 \neq z_2$. Let $\sigma_1 = \dfrac{z_1 - w_1}{w_2 - w_1}$ and $\sigma_2 = \dfrac{z_2 - w_2}{w_3 - w_2}$. Then $\operatorname{Re}\sigma_2 \geq \operatorname{Re}\sigma_1$.

Proof: First, $g(w) - f(w) = \dfrac{1}{3}\dfrac{w^2 + 3 - 2\sqrt{3 + w^2}}{w^2 - 1}$ is analytic in D which implies that $\operatorname{Re}(g(w) - f(w))$ is a harmonic function in D, so we my apply Proposition 1. Now $\lim\limits_{w\to\infty}(g(w) - f(w)) = \dfrac{1}{3}\lim\limits_{w\to\infty}\dfrac{w^2 + 3 - 2\sqrt{3 + w^2}}{w^2 - 1} = \dfrac{1}{3} \geq 0$. Thus $\liminf\limits_{k\to\infty}\operatorname{Re}(f(a_k) - g(a_k)) \geq 0$ for any sequence $\{a_k\}$ in D converging to ∞. Also, as $w \to \partial(D)$, $g(w) - f(w) \to \dfrac{1}{3}\dfrac{-t^2 + 3 \pm 2i\sqrt{t^2 - 3}}{-t^2 - 1} = \dfrac{1}{3}\dfrac{t^2 - 3 \pm 2i\sqrt{t^2 - 3}}{t^2 + 1}$. Then $\operatorname{Re}(g(w) - f(w)) \to \dfrac{1}{3}\dfrac{t^2 - 3}{t^2 + 1} \geq 0$ since $t^2 \geq 3$. It follows that $\liminf\limits_{k\to\infty}\operatorname{Re}(f(a_k) - g(a_k)) \geq 0$ for any sequence $\{a_k\}$ in D converging to $\partial(D)$. By Proposition 1, $\operatorname{Re}(\sigma_2(w) - \sigma_1(w)) \geq 0$, $w \in D$. For $w \in \partial(D)$, by (7) and (5), $\sigma_2 = -i\dfrac{it + 1}{2t - \sqrt{t^2 - 3}}$, which implies that

$$\sigma_2 - \sigma_1 = \frac{1}{3}\frac{-t^2 + 3 + 2i\sqrt{t^2 - 3}}{-t^2 - 1}. \qquad ((11))$$

By what was just proved, $\operatorname{Re}(\sigma_2(w) - \sigma_1(w)) \geq 0$, $w \in \partial(D)$.

Note: The example below shows that is possible to have $\operatorname{Re}\sigma_2 = \operatorname{Re}\sigma_1$. In fact, below we have $\sigma_1 = \sigma_2$.

Example: Let $w_1 = -1$, $w_2 = \sqrt{3}i$, $w_3 = 1$, which implies that $w = \sqrt{3}i$ and the $\{w_k\}$ are the vertices of an equilateral triangle. Then $z_1 = z_2 = \dfrac{1}{\sqrt{3}}i$ and $\sigma_1 = \sigma_2 = \dfrac{1}{2} - \dfrac{1}{6}i\sqrt{3}$.

It is natural to ask whether the example above gives essentially the only case when $\sigma_1 = \sigma_2$.

Theorem 4: $\sigma_1 = \sigma_2 \iff w_1, w_2, w_3$ are the vertices of an equilateral triangle which contains no vertical line segment.

Proof: $w = \pm 1 \Rightarrow w_2 = w_3$ or $w_2 = w_1$, in which case the ratios are not defined. Thus we may assume that $w \neq \pm 1$. For $w \in D$, $\sigma_1(w) = \sigma_2(w) \iff f(w) - g(w) = -\dfrac{1}{3}\dfrac{w^2 - 2\sqrt{3 + w^2} + 3}{(-1 + w)(w + 1)} = 0 \iff$

$w^2 - 2\sqrt{3 + w^2} + 3 = 0 \iff w = \pm i\sqrt{3} \iff \{w_1, w_2, w_3\} = \left\{-w_3, \pm\sqrt{3}iw_3, w_3\right\}$, which are easily seen to be the vertices of an equilateral triangle. For $w \in \partial(D)$, by (11), $\sigma_1(w) = \sigma_2(w) \iff$

$\frac{1}{3}\frac{-t^2+3+2i\sqrt{t^2-3}}{-t^2-1} = 0, w = ti, |t| \geq \sqrt{3}$. That yields $t = \pm\sqrt{3}$, which gives $w = \pm i\sqrt{3}$ as above. We can also assume that the triangle formed by w_1, w_2, w_3 contains no vertcial line segment, since the ratios are not defined in that case either.

Theorem 5: σ_1 or σ_2 are real if and only if w_1, w_2, and w_3 are collinear.

Proof: Suppose first that $\sigma_1(w)$ is real, $w \in D$. Then $f(w)$ is real, or $w + 3 - \sqrt{3+w^2} = k(w+1), k \in \Re$, which implies, after some simplification, that $(k^2-2k)w^2 + 2(1-k)(3-k)w + k^2 - 6k + 6 = 0$. The discriminant of this quadratic equation is $4(1-k)^2(3-k)^2 - 4(k^2-2k)(k^2-6k+6) = 4\left(2k-3\right)^2 \geq 0$ since $k \in \Re$. Hence w is real. Now if w is real, then for the ratios to exist, $w = \frac{w_2}{w_3}$ must be a positive real number, which we again denote by k. But then $w_1 = -kw_2$ and $w_3 = kw_2$. It is then easy to show that the set of points $\{-kw_2, w_2, kw_2\}$ must be collinear. If $\sigma_1(w)$ is real, $w \in \partial(D)$, then by (10) and (9), $-2t + \sqrt{t^2-3} = 0$, which has no real solutions. If σ_2 is real, we can proceed in the same fashion, or just use (5) to show that σ_1 is real.

Remark: One can easily extend the definition of complex ratios given in this paper to functions of the form $p(z) = (z-w_1)^{m_1}(z-w_2)^{m_2}(z-w_3)^{m_3}$, where m_1, m_2, and m_3 are given positive real numbers. This is discussed in [4] for all real w_j.

3 References

(1) Peter Andrews, Where not to find the critical points of a polynomial-variation on a Putnam theme, Amer. Math. Monthly 102(1995) 155–158.

(2) Sheldon Axler, Paul Bourdon, and Wade Ramey, Harmonic Function Theory, 2001, Springer Verlag, New York, Inc.

(3) Alan Horwitz, On the Ratio Vectors of Polynomials, Journal of Mathematical Analysis and Applications 205(1997), 568-576.

(4) Alan Horwitz, Ratio vectors of polynomial–like functions, preprint.

(5) Gideon Peyser, On the roots of the derivative of a polynomial with real roots, Amer. Math. Monthly 74(1967), 1102–1104.

(6) Walter Rudin, Real and Complex Analysis, 2nd ed., 1974, McGraw–Hill.